U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 3 - Year 11 - Academic Year 1999-2000

Gene A. Berg, Editor

1/3/11. We define the *repetition number* of a positive integer n to be the number of distinct digits of n when written in base 10. Prove that each positive integer has a multiple that has a repetition number less than or equal to 2.

Solution 1 by Peter Ruse (12/NY): Since there are finitely many possible remainders on division by n, two of the numbers $1, 11, 111, \ldots,$ must have the same remainder. Their difference, a number of the form $111...100...0$, is a multiple of n and has repetition number of 2.

Solution 2 by Michael Catlin (10/IN): We can express $\frac{1}{n}$ as the repeating decimal

$$0.b_1b_2b_3...b_k\overline{a_1a_2...a_m}$$

Then, $\frac{1}{n} = \frac{(10^m - 1)(b_1b_2\ldots b_k) + (a_1a_2\ldots a_m)}{10^k(10^m - 1)}$.

Thus, n divides $10^k(10^m - 1)$, a number with only the digits 9 and 0.

2/3/11. Let a be a positive real number, n a positive integer, and define the *power tower* $a\uparrow n$ recursively with $a\uparrow 1 = a$, and $a\uparrow(i+1) = a^{(a\uparrow i)}$ for $i = 1, 2, 3, \ldots$. For example, we have $4\uparrow 3 = 4^{(4^4)} = 4^{256}$, a number which has 155 digits. For each positive integer k, let x_k denote the unique positive real number solution of the equation $x\uparrow k = 10\uparrow(k+1)$. Which is larger: x_{42} or x_{43}?

Solution 1 by Lisa Leung (9/MD): By definition, $x_k\uparrow k = 10\uparrow(k+1)$, where both x_k and 10 are positive real numbers. Thus, $x_k > 10$ for all k.

$$x_{42}\uparrow 43 = x_{42}^{(x_{42}\uparrow 42)} = x_{42}^{10\uparrow 43} > 10^{10\uparrow 43} = 10\uparrow 44 = x_{43}\uparrow 43.$$

Thus, x_{42} is greater than x_{43}.

Editor's comments: This interesting problem was suggested by Roger Pinkham of Stevens Institute of Technology.

3/3/11. Suppose that the 32 computers in a certain network are numbered with the 5-bit integers 00000, 00001, 00010, ..., 11111 (bit is short for binary digit). Suppose that there is a one-way connection from computer A to computer B if and only if A and B share four of their bits with the remaining bit being 0 at A and 1 at B. (For example, 10101 can send messages to 11101 and to 10111.) We say that a computer is at level k in the network if it has exactly k 1's in its label ($k = 0, 1, 2, ..., 5$). Suppose further that we know that 12 computers, three at each of the levels 1, 2, 3, and 4, are malfunctioning, but we do not know which ones. Can we still be sure that we can send a message from 00000 to 11111?

Solution 1 by Jennifer Balakrishnan (10/GU):
Computer at level 0: 00000
Computers at level 1: 00001, 00010, 00100, 01000, 10000
Computers at level 2: 11000, 10100, 10010, 10001, 01100, 01010, 01001, 00110, 00101, 00011
Computers at level 3: 00111, 01011, 01101, 01110, 10011, 10101, 10110, 11001, 11010, 11100
Computers at level 4: 11110, 11101, 11011, 10111, 01111
Computer at level 5: 11111

Assuming that "malfunctioning" means "will not accept information," then we cannot be sure that we can send a message from 00000 to 11111.

At level 0, the computer can send the information I to any of 5 computers. At level 1, each of these five computers can send I to four computers on level 2. At level 2 each computer can send I to three computers on level 3. On level 3 each computer can send I to two computers on level 4. On level 4 each computer can send I to 1 computer on level 5.

Suppose at level 1, three computers are affected. I still can be sent to the other two computers at level 1. These two computers, P and Q, can normally send to seven level 2 computers; assume among these seven that the three that P does not send to are malfunctioning, so Q sends I to one good level 3 computer, and P sends to this one good level 3 computer (call it R) and three other good level 3 computers (call them S, T, and U). R sends to three level 3 computers and together, R, S, T, and U send to six level 3 computers. Assume that among these six, the computers R does not send to are malfunctioning. Now on level four, only the three computers which are "grand-children" of R will be sent I. Assume these three malfunction. So the information stops here on level four and cannot be transmitted to level 5.

Solution 2 by Asher Walkover (12/NY): Rather than limit my answer to the case with 2^5 computers in the network, I will prove a more general result. Consider any system of 2^n computers (where $4 < n$), each computer given a binary name, networked as described. I shall prove that it will always be possible to select n-2 computers on levels 1, 2, $n-2$, and $n-1$, such that, if they are malfunctioning, a message sent from the level 0 computer *(000...)* will not get through to the level n computer *(111...)*.

On level 1, each computer has a single 1 in its name. Its name is n digits long so there are n possible ways a level 1 computer can be named, which means there are n computers on level 1. Since there are only n level 1 computers, of which $n-2$ are malfunctioning, only two level 1 computers will be working. Let those two computers be the computer with 1 as first digit (call it computer A) and the computer with 1 as the second digit (call it computer B).

Since only A and B are functioning at level 1, all level 2 computers who receive the message receive it from A or B or both. From computer B we can access all $n-1$ level 2 computers with a 1 as their second digit and one more 1 somewhere else in the name. Consider, however, that one of these computers (the one whose first two digits are 1's, the rest 0's) can be reached by both computers A and B. Thus there are only $n-2$ level 2 computers which are reachable only by computer B. Let those $n-2$ level 2 computers be malfunctioning. It follows, therefore, that all functioning level 2 computers which can receive a message are the ones that are linked to computer A. A has a 1 as the first digit, and so must all the level 2 computers linked to A. Thus, the only computers on level 2 that receive the message are the computers with 1 as the first digit.

After the message has traveled through all the levels (if any exist) between level 2 and level $, n-2$, all level $n-2,$ computers with a 1 as first digit may have the message. If the computer does not have a 1 as the first digit, it cannot have the message, since the only level 2 computers to receive and pass on the message had a 1 as first digit. Thus, all level $n-2$ computers who have the message have the first two digits either 11 or 10. Consider the later case. In the remaining $, n-2$ digits of those computers' names, there is only one 0. There are $n-2$ such computers (computers on level $n-2$ that begin with 10). If these $n-2$ computers are all malfunctioning, then the only level $n-2$ computers to receive the message will be those whose first digits are 11. Since all the level $n-2$ computers with the message begin with 11, they can only send a message to a level $n-1$ computer that begins with a 11.

On level $n-1$, each computer has $n-1$, 1's and one 0. There are n such computers. $n-2$ computers are malfunctioning at that level, so only two are working. However, if we let the two working computers be the ones whose first digits are 10 and 01 (with all the rest of the digits 1's) then neither will be able to receive the message froma level $n-2$ computer that begins with 11. But the level $n-2$ computers that begin with 11 are the only functioning level $n-2$ computers with the message, so neither one of the working level $n-1$ computers will receive the message, and it will not be passed on to the computer on level n.

If we let $n = 5$, then we have 3 computers malfunctioning on levels 1, 2, 3, and 4, as stipulated, so we see it is not necessarily possible that a message sent from the level 0 computer will reach the level n computer.

Editor's comments: This problem is a special case of a theorem of our Problem Editor, Bela Bajnok of Gettysburg College, and Shahriar Shahriari of Pomona College. The general result, which appeared in the *Journal of Combinatorial Theory, Series A*, in 1996, says that it is possible to remove $n-2$ nodes at each of the levels 1, 2, ..., $n-1$, of the Boolean lattice of order n (our network) in such a way that the source gets disconnected from the target, and that this is not possible if we remove $n-3$ at each level.

4/3/11. We say a triangle in the coordinate plane is *integral* if its three vertices have integer coordinates and if its three sides have integer lengths.
(a) Find an integral triangle with perimeter of 42.
(b) Is there an integral triangle with perimeter of 43?

Solution 1 by Clayton Myers (11/NJ): (a) The triangle with vertices (0,0), (14,0), and (9,12) is an integral triangle with perimeter 42 and sides of lengths 13, 14, and 15.

(b) Since an integral triangle can be translated horizontally and vertically in integral steps without effect, let us assume that a hypothetical integral triangle with perimeter 43 has vertices (0,0), (x_1,y_1), and (x_2,y_2). Now, since all side lengths are integers and their sum is 43, we know that either all three sides are odd, or two are even and one is odd. Therefore, since their squares must have the same parity, the sum of their squares must be odd. This number can be expressed as

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 = 2(x_1^2 + y_1^2 + x_2^2 + y_2^2 - x_1x_2 - y_1y_2)$$

which is even. This contradiction means that our assumption that an integral triangle with perimeter 43 exists is false.

Solution 2 by Robert Kotredes (12/ME): (a) The triangle with vertices (0,0), (12,9), and (12,16) has sides of length 20, 15, and 7, so is an integral triangle with perimeter 42.

(b) By Pick's Theorem, the area of any convex polygon with integer coordinates is given by $A = I + \left(\frac{B}{2}\right) - 1$, where I is the number of integer points interior to the polygon, and B is the number of integer points along its border. This is a rational number.

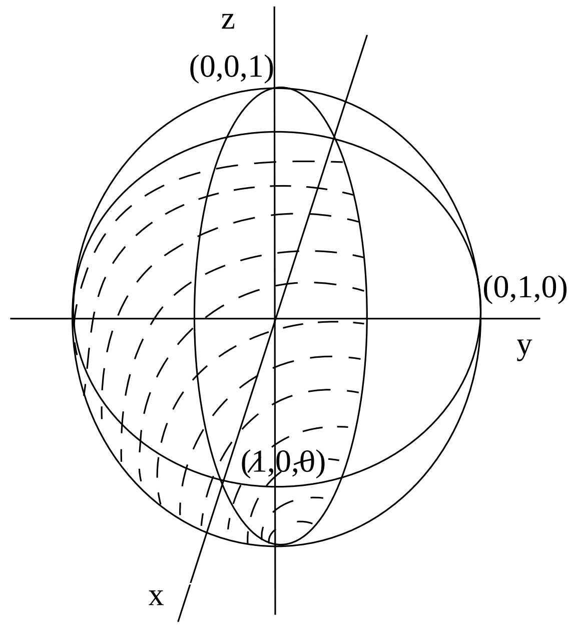
By Heron's Formula, the area of a triangle with sides of lengths *a, b,* and *c,* is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a + b + c)/2$. For any triangle with perimeter 43 this becomes $A = \sqrt{\frac{43(43-2a)(43-2b)(43-2c)}{16}} = \frac{1}{4}\sqrt{43(43-2a)(43-2b)(43-2c)}$.

Observe 43 is prime. Also *(43 -2a), (43 - 2b),* and *(43 - 2c)* are positive integers smaller that 43, so don't have 43 as a factor. The area of the triangle is therefore $N \cdot \sqrt{43}$ where N is a number. Hence the area is an irrational number. So if there were an integral triangle with perimeter 43 it would have both an irrational and a rational area, a contradiction. Thus, there is no integral triangle with perimeter 43.

Editor's comments: This problem was derived from a problem proposal by Robert Ward of NSA. He came upon the problem as a result of his work with the "Ask Dr. Math" program where he serves as a consultant to deal with questions submitted by high school students. For more "Ask Dr. Math" details see http://forum.swarthmore.edu/dr.math.

5/3/11. We say that a finite set of points is *well scattered* on the surface of a sphere if every open hemisphere (half the surface of the sphere without its boundary) contains at least one of the points. The set { (1,0,0), (0,1,0), (0,0,1) } is not well scattered on the unit sphere (the sphere of radius 1 centered at the origin), but if you add the correct point *P* it becomes well scattered. Find, with proof, all possible points *P* that would make the set well scattered.

Solution 1 by Alexey Gorshkov (12/MA):
Answer: All points on the sphere that have all three of their coordinates negative. That is, one eighth of the unit sphere without its boundary. (See Figure at right.)



Proof (it consists of two sub-proofs):

Proof that all points not belonging to the region do not make the set well-scattered: Since the point is not in the region described above, at least one of its coordinates is positive or zero.Then the open hemisphere that has all the points with this coordinate negative will not contain any of the four points, which proves that all points not belonging to the region do not make the set well scattered.

Proof that all points belonging to the region do make the set well scattered: Let's suppose we found an open hemisphere *H* that does not contain the three given points and some point *P* in the region described above in the answer. Since all three coordinates of *P* are negative, the point *P'* diametrically opposite to *P* lies in the "triangular" region determined by the three points given initially. Since any open hemisphere that contains a point in this "triangular" region must contain at least one of its vertices[(1)], the open hemisphere *H* does not contain the point *P'*. However two diametrically opposite points (*P* and *P'*) can be both outside an open hemisphere only if they lie on its boundary, which cannot be the case[(2)]. So *H* does not exist. So all the points belonging to the region described in the answer do make the case well scattered.

[(1)]If this is not obvious, then here is a proof. Let's suppose there is an open hemisphere *G* containing a point *X* inside the "triangular" region and not containing any of the three "vertices". Since *G* does not contain any of the "vertices", it does not contain any points of the sides (this is because two circles of radii equal to that of the sphere either coincide or intersect in two diametrically opposite points). Since *G* does not contain any points on the boundary of the "triangular" region and does not contain a point inside, it must be completely inside, which is impossible.

[(2)]If this is not obvious, then the proof is identical to the one in the previous paragraph with the only change that *G* is a closed hemisphere (so that it contains *P'*), which does not make any difference in the proof.

Solution 2 by Daniel Kane (9/WI): The set of points that would make it well scattered are the points inside the spherical triangle with vertices (-1,0,0), (0,-1,0), and (0,0,-1); call this triangle *A*. You can see that any point outside of *A* would not make the set well scattered by looking at the hemispheres determined by the great circles through (1,0,0) and (0,1,0), (1,0,0) and (0,0,1), and (0,1,0) and (0,0,1).

Any point within the spherical triangle *A* is diametrically opposite a point inside the spherical triangle (1,0,0), (0,1,0),(0,0,1). Therefore, the tetrahedron determined by the four points in the set would contain the center of the sphere. Given any four points on the same hemisphere (of the ball) the tetrahedron they determine would also be contained in that hemisphere. Therefore if the tetrahedron contains the center of the sphere, the points are well scattered.

Therefore, given P is not in the spherical triangle *A*, the points are not well scattered, and given that point *P* is in triangle *A*, {(1,0,0), (0,1,0), (0,0,1), *P*} is well scattered.

Editor's comments: This attractive problem is based on the work of Andrew Lenard, a retired Hungarian mathematics professor at Indiana University.